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Path integral and gravitational radiation damping†

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Abstract. The energy loss of a gravitationally bound, quantum mechanical matter system due to gravitational radiation damping is calculated in first approximation using Feynman's path integral formalism. The classical limit is discussed, confirming the classical quadrupole radiation formula.

1. Introduction

In a previous paper (Schäfer and Dehnen 1980) Einstein's classical quadrupole radiation formula was established by calculating the spontaneous transition probabilities of a bound quantum mechanical matter system with respect to gravitational radiation from Einstein's absorption coefficients, using the method of thermal equilibrium between the matter system and the radiation field. The absorption coefficients were obtained using Dirac's usual time-dependent perturbation theory without quantisation of the radiation field.

In this paper we give a full field theoretical derivation of the energy loss of a gravitationally bound, quantum mechanical matter system due to gravitational radiation. First we deduce the classical Lagrangian of a gravitationally bound matter system interacting with the gravitational wave field on the quadrupole approximation level, and then we quantise the whole according to Feynman's path integral formalism. In this framework the spontaneous transition amplitude for the decay of the quantised matter system is calculated. The result confirms again the classical quadrupole radiation formula

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2}{45}\frac{f}{c^5}\sum_j \omega_j^6 Q^{ab}(\omega_j)Q^*_{ab}(\omega_j) \tag{1}$$

(f is the Newtonian gravitational constant, and $Q_{ab}(\omega)$ the Fourier transform of the mass quadrupole tensor) in the sense of the correspondence principle.

As in our previous paper, the difficulties of a consistent approximative integration of the inhomogeneous classical field equations of gravitation are avoided, and fewer approximation steps are necessary for calculating the radiation damping than in the usual classical case. Beyond this, no use is made of any energy pseudotensor for the gravitational field, in contrast to our previous paper.

† This paper is based on an essay which received an honourable mention (1980) from the Gravity Research Foundation.

2. The model

In view of the quantum mechanical approach, we choose for the matter a finite system of point masses, the energy–momentum tensor of which takes the form†

$$T_{\mu\nu}=\sum_i \frac{m_i c^2}{\sqrt{-g}} g_{\mu\rho} g_{\nu\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{dx^4}\delta(\boldsymbol{x}-\boldsymbol{x}_{(i)}), \tag{2}$$

where m_i is the mass and $\boldsymbol{x}_{(i)}$ the space-like position of the ith particle ($d\tau^2=-ds^2$; the signature of $g_{\mu\nu}=(+,+,+,-)$). Einstein's field equations for the metric $g_{\mu\nu}$ read

$$R_{\mu\nu}=(8\pi f/c^4)(T_{\mu\nu}-\tfrac{1}{2}Tg_{\mu\nu}) \tag{3}$$

and for the metric we make the ansatz

$$g_{\mu\nu}=\eta_{\mu\nu}+u_{\mu\nu}+h_{\mu\nu}+v_{\mu\nu}. \tag{4}$$

Here $\eta_{\mu\nu}$ is the Minkowski metric in Cartesian coordinates and its perturbations $u_{\mu\nu}$, $h_{\mu\nu}$ and $v_{\mu\nu}$ are considered as small compared with $\eta_{\mu\nu}$.

The quantity

$$u_{\mu\nu}=-(2\Phi/c^2)\delta_{\mu\nu} \tag{5}$$

takes into account the instantaneous Newtonian gravitational field of the point-particle system, the potential Φ of which is given by

$$\Phi=-f\sum_i m_i/|\boldsymbol{x}-\boldsymbol{x}_{(i)}|. \tag{5a}$$

Evidently $u_{\mu\nu}$ is of the order of c^{-2} and fulfils to this order the de Donder condition‡

$$u_\mu{}^\alpha{}_{|\alpha}-\tfrac{1}{2}u_\alpha{}^\alpha{}_{|\mu}=0. \tag{6}$$

The field $h_{\mu\nu}$ represents the gravitational wave field with the properties

$$h_{\mu 4}=0, \qquad h_a{}^a=0, \qquad h_m{}^a{}_{|a}=0 \tag{7}$$

(TT-gauge), and fulfils the inhomogeneous wave equation, from (2) and (3) with the use of (5) and (5a):

$$h^{ab|\alpha}{}_{|\alpha}=-\frac{32\pi f}{c^4}\int\sum_i\left(\frac{m_i}{2}\dot{x}^a_{(i)}\dot{x}^b_{(i)}-\frac{1}{4}\sum_{j(\neq i)}\frac{f m_i m_j (x^a_{(i)}-x^a_{(j)})(x^b_{(i)}-x^b_{(j)})}{|\boldsymbol{x}_{(i)}-\boldsymbol{x}_{(j)}|^3}\right)^{\mathrm{TT}} \times\exp[\mathrm{i}\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}_{(i)})]\frac{d^3k}{(2\pi)^3}. \tag{8}$$

Here TT means the transverse-traceless part obtained by projection with the projection operator $P_{ab}=\delta_{ab}-n_a n_b$, $\boldsymbol{n}=\boldsymbol{k}/k$ ($k=|\boldsymbol{k}|$), cf Misner *et al* (1973). For the derivation of (8) we have restricted ourselves on the right-hand side to the *leading* TT-*projected* terms with respect to powers of c^{-1} ($\dot{x}^a=dx^a/dt$).

† Greek indices run from 1 to 4 ($x^4=ct$) and Latin indices from 1 to 3.

‡ Raising and lowering of indices is performed by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ respectively. $|\mu$ means the partial derivative with respect to the coordinate x^μ.

In view of the comparison with the later quantum mechanical calculation, we give here the retarded solution of (8); it takes the form

$$h^{ab}(\boldsymbol{x}, t) = \frac{32\pi f}{c^4} \int \left(\sum_i \frac{m_i}{2} \dot{x}^a_{(i)} \dot{x}^b_{(i)} - \frac{1}{4} \sum_{i \neq j} \frac{f m_i m_j (x^a_{(i)} - x^a_{(j)})(x^b_{(i)} - x^b_{(j)})}{|\boldsymbol{x}_{(i)} - \boldsymbol{x}_{(j)}|^3} \right)^{\mathrm{TT}}_{t'} \times \frac{\exp(\mathrm{i}\boldsymbol{kx}) \exp[-\mathrm{i}\omega(t-t')]}{k^2 - (\omega + \mathrm{i}\varepsilon)^2/c^2} \frac{\mathrm{d}^3k \, \mathrm{d}\omega \, \mathrm{d}t'}{(2\pi)^4}. \tag{8a}$$

Evidently only the reciprocal wavelength $k \simeq \omega_0/c$ contributes essentially to the integral, where ω_0 is a typical frequency of the matter system. The leading term of h_{ab} with respect to powers of c^{-1} was found, when going over from (8) to (8a), by setting $\boldsymbol{kx}_{(i)} \to 0$ (wavelength large compared with the linear dimension of the matter system, quadrupole approximation). This term is of the order of c^{-4}.

Because of the 'zero' boundary conditions (no solution of the homogeneous wave equation present) in equation (16a) and the fact that those h^{ab} contribute most of the path integral in § 4 which are of the same order as (8a) (see the Appendix), it is sufficient to consider the order of magnitude of h_{ab} in the following as c^{-4}†.

Finally, the term $v_{\mu\nu}$ in (4) describes all perturbations of the Newtonian potential $u_{\mu\nu}$ and the wave field $h_{\mu\nu}$. As for $h_{\mu\nu}$, we demand for $u_{\mu\nu} + v_{\mu\nu}$ the de Donder condition exactly. Then from (2) and (3) we obtain, in view of (4)–(7), taking into account all terms up to the order of $c^{-2}h_{ab}$ (leading terms containing h_{ab}), the following differential equation for v_{44}:

$$v_{44}{}^{|\alpha}{}_{|\alpha} = h^{ab} u_{44|a|b}, \tag{9}$$

which results from the homogeneous part of (3) only. The terms of order up to c^{-6}, but independent of h_{ab}, are omitted because they give rise to a fine-structure of the matter system only. The differential equations for the other components of $v_{\mu\nu}$ are not required with respect to the Lagrangian approach in § 3.

In view of the approximation in equation (9), it makes sense only if we restrict ourselves furthermore to the leading term with respect to powers of c^{-1}, neglecting the retardation of h^{ab} over the material system by considering h^{ab} as a function of t alone. Then we obtain from (9), with regard to (5) and (5a), under appropriate limiting procedures and the boundary condition h^{ab}, $u_{44} \to 0 \Rightarrow v_{44} \to 0$, the near-zone solution (for details see Schäfer and Dehnen (1980)):

$$v_{44} = -(f/c^2) h_{ab}(t) \sum_i m_i (x^a - x^a_{(i)})(x^b - x^b_{(i)})/|\boldsymbol{x} - \boldsymbol{x}_{(i)}|^3. \tag{10}$$

3. The Lagrangian

The action functional of the *total* system has, according to (2) and (3), the form

$$S = -\sum_i m_i c \int \mathrm{d}\tau_i + \frac{c^4}{16\pi f} \iint R\sqrt{-g}\, \mathrm{d}^3x \, \mathrm{d}t. \tag{11}$$

Inserting (4), we obtain from the material term, in the same approximation as in

† We thank the anonymous referee for bringing this to our attention.

equation (9), the Lagrangian ($S=\int L\,\mathrm{d}t$)

$$L_{\mathrm{m}}=-\sum_i m_i c^2[1-\tfrac{1}{2}(u_{44}+v_{44})-\tfrac{1}{2}(\eta_{ab}+h_{ab})\dot{x}^a_{(i)}\dot{x}^b_{(i)}/c^2] \tag{12a}$$

and, after elimination of the second derivatives of the metric† from the gravitational term, the Lagrangian

$$L_{\mathrm{g}}=-\frac{c^4}{64\pi f}\int h_{ab|\mu}h^{ab|\mu}\,\mathrm{d}^3x-\sum_i m_i c^2\tfrac{1}{4}(u_{44}+v_{44}) \tag{12b}$$

using the properties (5)–(7) and (9). In (12b) the first expression on the right-hand side represents the leading term of the free radiation field, whereas the second one is of the same constitution as (12a). Now we rearrange the terms of the total Lagrangian $L=L_{\mathrm{m}}+L_{\mathrm{g}}$ in a natural manner as $L=L_{\mathrm{mat}}+L_{\mathrm{int}}+L_{\mathrm{rad}}$ with

$$L_{\mathrm{mat}}=\sum_i\frac{m_i}{2}\dot{x}^a_{(i)}\dot{x}^b_{(i)}\eta_{ab}+\frac{1}{2}\sum_{i\neq j}\frac{fm_im_j}{|\boldsymbol{x}_{(i)}-\boldsymbol{x}_{(j)}|}, \tag{13a}$$

$$L_{\mathrm{int}}=h_{ab}(t)\bigg(\sum_i\frac{m_i}{2}\dot{x}^a_{(i)}\dot{x}^b_{(i)}-\frac{1}{4}\sum_{i\neq j}\frac{fm_im_j(x^a_{(i)}-x^a_{(j)})(x^b_{(i)}-x^b_{(j)})}{|\boldsymbol{x}_{(i)}-\boldsymbol{x}_{(j)}|^3}\bigg), \tag{13b}$$

$$L_{\mathrm{rad}}=-\frac{c^4}{64\pi f}\int h_{ab|\mu}h^{ab|\mu}\,\mathrm{d}^3x, \tag{13c}$$

where (5), (5a) and (10) are inserted and the irrelevant rest- and self-energies of the point masses are dropped.

Evidently L_{mat} and L_{rad} represent the leading Lagrangians of the free matter system and the free radiation field respectively, whereas L_{int} is the leading interaction Lagrangian between matter and radiation. Using the Hamiltonian H_{mat} of the undisturbed material system, equation (13b) can be written with the help of the Poisson-bracket formalism within the order of $h_{ab}(t)$ as

$$\begin{aligned}L_{\mathrm{int}}&=\tfrac{1}{4}h_{ab}(t)\sum_i m_i[H_{\mathrm{mat}},[H_{\mathrm{mat}},x^a_{(i)}x^b_{(i)}]]\\&=\tfrac{1}{12}h_{ab}(t)\ddot{Q}^{ab}\end{aligned} \tag{14}$$

with the traceless mass quadrupole tensor (remember $h_a{}^a=0$)

$$Q^{ab}=\sum_i m_i(3x^a_{(i)}x^b_{(i)}-\boldsymbol{x}^2_{(i)}\eta^{ab}). \tag{14a}$$

Finally we note that higher correction terms (counted in powers of c^{-1}) would be added to the expressions (13a)–(13c) corresponding to their character (pure matter, interaction, pure radiation).

4. The path integral

According to Feynman's path integral quantisation, the transition amplitude of the *total*

† This procedure does not destroy the gauge invariance of the action functional (see Weyl 1970) and coincides with the reconstruction of the Lagrangian of general relativity by Gibbons and Hawking (1977); see also Weinberg (1979).

system from an initial to a final state is defined by

$$A = \iint_{t_i}^{t_f} \exp\left(\frac{i}{\hbar}[S_{\text{mat}}(x) + S_{\text{int}}(x, h) + S_{\text{rad}}(h)]\right) \mathscr{D}x\mathscr{D}h, \tag{15}$$

where t_i and t_f mean the times of the initial and final state respectively. The action functionals S_{mat}, S_{int} and S_{rad} correspond to the Lagrangians (13*a*), (13*b*) or (14), and (13*c*).

Because our aim is the calculation of the *energy loss of the matter system* by gravitational radiation, we consider its spontaneous decay. For this we first evaluate the transition amplitude (15) for the 0 to 0 graviton transition. Following Feynman and Hibbs (1965), a straightforward calculation yields

$$A(0, 0) = \int_{t_i}^{t_f} \exp\left(\frac{i}{\hbar}[S_{\text{mat}}(x) + I(x)]\right) \mathscr{D}x, \tag{16}$$

where

$$I(x) = \frac{\hbar}{i} \ln \int_0^0 \exp\left(\frac{i}{\hbar}[S_{\text{int}}(x, h) + S_{\text{rad}}(h)]\right) \mathscr{D}h \tag{16a}$$

is given with the use of (14) and (13*c*) by

$$I(x) = i\frac{f}{90\pi c^5} \int_0^\infty \iint_{-\infty}^{+\infty} \dddot{Q}^{ab}(t)\dddot{Q}_{ab}(t') \exp(-i\omega|t-t'|)\, dt\, dt'\, \omega\, d\omega. \tag{16b}$$

In view of the saddle point method, the dominant contribution of the path integral (16*a*) comes from such values of h_{ab} which satisfy the classical equation of motion. From this it is evident that higher correction terms in (13*c*) and (14) do not disturb the c^{-5} order of the result (16*b*), which is already the order of the final result (21).

In obtaining equation (16*b*) the tracelessness of Q^{ab} is used explicitly, as well as the assumption that the wavelength of the radiation is large compared with the size of the matter system (quadrupole approximation). The latter is not in contradiction to the ω-integration in (16*b*) because, as we shall see later, the high frequencies do not contribute to the integral.

Then the amplitude for the matter transition from state M to M becomes, from (16) and (16*b*),

$$A_{MM}(0, 0) = \iiint_{t_i}^{t_f} \psi_M^*(x_f) \exp\left(\frac{i}{\hbar}[S_{\text{mat}}(x) + I(x)]\right) \psi_M(x_i) \mathscr{D}x(t)\, dx_i\, dx_f \tag{17}$$

($\psi_M(x)$ is the wavefunction of the state M). Expansion with respect to I yields in the first-order approximation for an energy eigenstate M with energy E_M

$$A_{MM}(0, 0) = \exp[-(i/\hbar)(E_M + \Delta E_M)T], \qquad T = t_f - t_i, \tag{18}$$

where

$$\text{Im}\, \Delta E_M = -\frac{f}{45c^5} \sum_N \int_0^\infty \langle M|\dddot{Q}^{ab}|N\rangle\langle N|\dddot{Q}_{ab}|M\rangle \delta\left(\frac{E_M - E_N}{\hbar} - \omega\right) \omega\, d\omega \tag{18a}$$

is valid, which has its origin in the imaginary part of I (cf (16*b*)) only. The δ- function means that the ω- integration does not disturb the quadrupole approximation. The real part of ΔE_M diverges but has no significance for the *decay* of the matter system. The probability for this is given with regard to (18) by

$$\exp(2\,\mathrm{Im}\,\Delta E_M T/\hbar). \tag{19}$$

Accordingly the probability per time unit for spontaneous transition from the energy eigenstate M into all *lower* energy eigenstates N has, using (18a), the form ($\hbar\omega_{MN} = E_M - E_N$)

$$\Gamma_{M\to\sum N} = \frac{2f}{45c^5}\frac{1}{\hbar}\sum_N \omega_{MN}^5 \langle M|Q^{ab}|N\rangle\langle N|Q_{ab}|M\rangle. \tag{20}$$

We note explicitly that higher correction terms in the Lagrangian (13a) would give rise only to relativistic corrections in the eigenstates N, M and their eigenvalues E_N, E_M respectively (fine-structure) and do not disturb the structure of the formulae (18a) and (20). Nevertheless, within a consistent approximation these correction terms must be neglected.

In consequence of the transition probability (20), the total energy loss per time unit of the material system by its spontaneous decay with respect to the gravitational interaction reads

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2f}{45c^5}\sum_N \omega_{MN}^6 \langle M|Q^{ab}|N\rangle\langle N|Q_{ab}|M\rangle. \tag{21}$$

This result is in exact agreement with the classical equation (1) in the sense of the correspondence principle, whereby this principle is applied to the non-relativistic matter only and not to the gravitational wave field.

Appendix

According to the saddle point method the path integral (16a) can be approximately written as

$$I(x) \simeq S_{\mathrm{int}}(x, h_{\mathrm{class}}) + S_{\mathrm{rad}}(h_{\mathrm{class}}). \tag{A1}$$

Here h_{class} is a 'classical' solution of the classical equations of motion, following from (13c) and (14) with the help of Hamilton's principle, which are identical with equation (8) in the quadrupole approximation. Because the special boundary condition (the 0 to 0 graviton transition in equation (16a)) is connected with tunnelling or barrier penetration of gravitons, we are forced to take the solution of the 'classical' Euclidean (imaginary-time) equations of motion which vanishes at infinity, cf for example Gervais (1978). This solution is unique (cf DeWitt 1964) and is given, if analytically continued to the Minkowski metric, by the Feynman propagator in the following way:

$$h^{ab}_{\mathrm{class}}(\boldsymbol{x}, t) = \frac{8\pi f}{3c^4}\int \ddot{Q}^{TT}_{ab}(t')\frac{\exp(\mathrm{i}\boldsymbol{kx})\exp[-\mathrm{i}\omega(t-t')]}{k^2-\omega^2/c^2-\mathrm{i}\varepsilon}\frac{\mathrm{d}^3k\,\mathrm{d}\omega\,\mathrm{d}t'}{(2\pi)^4}. \tag{A2}$$

The difference from the classical solution (8a) consists only in the different Green functions in both equations. The complex valuedness of the Feynman propagator in (A2) is essential for the fact that the imaginary part of $I(x)$, which determines the final result (21) solely, does not vanish. If we insert (A2) into (A1) we obtain for $I(x)$, using the equations ($13c$) and (14), the 'approximate' expression

$$I(x) \simeq \frac{f}{90\pi^2 c^5} \iiint_{-\infty}^{+\infty} \int_0^{\infty} \dddot{Q}^{ab}(t)\dddot{Q}_{ab}(t') \frac{\omega^2 \exp[-\mathrm{i}\omega'(t-t')]}{\omega^2-\omega'^2-\mathrm{i}\varepsilon} \,\mathrm{d}\omega\, \mathrm{d}\omega'\, \mathrm{d}t\, \mathrm{d}t', \tag{A3}$$

which however, because of the relation

$$\exp(-\mathrm{i}\omega|t-t'|) = \frac{\mathrm{i}}{\pi}\omega \int_{-\infty}^{+\infty} \frac{\exp[-\mathrm{i}\omega'(t-t')]}{\omega'^2-\omega^2+\mathrm{i}\varepsilon}\, \mathrm{d}\omega',$$

is already identical to the exact solution ($16b$). Consequently the 'stationary' values of h_{ab}, according to (A2), determine the path integral completely in our case.

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